Theory of Inelastic Proton-Deuteron Scattering*

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Calculations are made of the small-angle inelastic proton-deuteron cross section near threshold. The theory assumes impulse approximation to relate the interaction of the incident proton with the target nucleons to the free nucleon-nucleon scattering amplitude. Effective-range theory is used to describe the final-state interaction of the target nucleons in the S state. The cross section at small angles is dominated by events which leave the target nucleons in the singlet S state. The cross section depends on three nucleon-nucleon parameters: the coefficients of the singlet S and triplet S terms in the cross section, Σ_s and Σ_t and the sum of the proton-neutron and proton-proton differential cross sections, $\sigma_{np}+\sigma_{pp}$. These parameters are determined at the laboratory angles of 5°, 10°, 15°, and 20° by making a least-squares fit to the experimental measurements of Stairs, Wilson, and Cooper in the preceding paper. This fit mainly determines Σ_s. This parameter is particularly sensitive to the isotopic spin zero nucleon-nucleon amplitudes and thus these values of Σ_s may be of use in future phase-shift analyses.

I. INTRODUCTION

THE scattering of high-energy protons from deuterium has been extensively studied in recent years with the aim of relating the proton-deuteron cross section directly to the fundamental nucleon-nucleon interaction. The elastic proton-deuteron cross section and polarization have been measured by Postma and Wilson[1] and studied in the impulse approximation by Kerman, McManus, and Thaler.[2] The quasi-free (often called quasi-elastic) proton-deuteron scattering has been studied experimentally by Kuckes and Wilson[3] and theoretically by Everett.[4] Quasi-free scattering is an inelastic process in which the incident proton interacts with one of the target nucleons essentially as if the latter were free. Two detectors are placed about 90° apart to detect the two outgoing particles and their energies are selected to insure that the third particle (the spectator nucleon) has nearly zero energy. Under these circumstances the theory says that the proton-deuteron cross section can be related directly to the free proton-nucleon cross section.

In this paper we discuss a different type of inelastic process, which we term slightly-inelastic scattering. By slightly-inelastic we mean an inelastic process in which the incident proton loses only a few MeV. In such a process a high-energy incident proton interacts with one of the target nucleons and transfers to it just enough energy to disintegrate the deuteron. Thus, the process is $p+d \rightarrow (n+p)+p$, where the final proton has the high energy. One is interested in the cross section $d\sigma/d\Omega_p dE'$, where Ω_p is the solid angle into which the high-energy proton is scattered and E' is the final energy of the high-energy proton. Experimentally one observes, at a fixed scattering angle, a spectrum of outgoing protons with energies a few MeV less than the elastically scattered protons. Because the resultant neutron-proton system has only a few MeV in its center-of-mass system, the amplitudes for the scattering of the incident proton from the target neutron and proton interfere, and so the slightly-inelastic cross section cannot be related directly to the free nucleon-nucleon cross sections.

In any inelastic proton-deuteron process the target nucleons go from the deuteron ground state, which is a triplet spin state, to a free scattering state in the continuum, which is a statistical mixture of triplet and singlet states. However, for small scattering angles where the momentum transfer, q, is small,[5] the probability that the deuteron makes a transition to a free triplet state is very small due to the orthogonality of the spatial part of the deuteron wave function and all triplet scattering state wave functions. This means that the scattering will involve predominantly triplet-singlet transitions in spite of the smaller *a priori* probability of finding a singlet state. Such a triplet-singlet transition involves only the spin-dependent parts of the nucleon-nucleon scattering matrix and, as we shall see, brings in the isotopic spin zero and one ($T=0$ and $T=1$) matrix elements with equal weight. This latter fact opens the possibility of using small-angle slightly-inelastic proton-deuteron scattering measurements to obtain valuable information about the $T=0$ part of the nucleon-nucleon scattering matrix. We discuss this in more detail in Sec. IV.

Small-angle slightly-inelastic proton-deuteron scattering is particularly suitable for theoretical treatment since the conditions are such as to allow reasonably accurate calculations to be made without highly involved computation. Because the incident particle has high energy the impulse approximation can be used to

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[1] H. Postma and R. Wilson, Phys. Rev. **121**, 1229 (1961).
[2] A. Kerman, H. McManus, and R. Thaler, Ann. Phys. (N. Y.) **8**, 551 (1959).
[3] A. Kuckes and R. Wilson, Phys. Rev. **121**, 1226 (1961).
[4] A. Everett, Phys. Rev. **126**, 831 (1962).

[5] In the case of 158-MeV incident protons considered in this paper, q is at most 1.0 F^{-1} at the maximum laboratory scattering angle of 20°. (Here, and throughout this paper, we use a system of units in which $\hbar=c=1$.)

relate the fundamental nucleon-nucleon interaction to the free-particle scattering matrix. This approximation is particularly good at small angles, where off-energy-shell corrections are small. Also, since the resultant neutron-proton system is left with only a small amount of energy in its center of mass, the final-state wave functions can be described accurately using potentials fitted to the low-energy nucleon-nucleon data. Furthermore, since the momentum transfer is small, the results are not sensitive to the detailed structure of the final-state wave functions at small distances.

The general theory of slightly-inelastic proton-deuteron scattering is presented in Sec. II. In Sec. III A calculations are made assuming that only S-wave neutron-proton scattering states are involved, using a square well potential fitted to the low-energy data. In Sec. III B the scattering to higher angular momentum states is considered. As the scattering angle increases this contribution increases and is comparable to the S-wave contribution at 15°.

The cross section for the reaction $n+d \rightarrow (n+n)+p$ in which the final proton has the high energy was calculated some years ago by Gluckstern and Bethe.[6] Their calculation used a particular form for the nucleon-nucleon potential and neglected both the factor $\exp(i\frac{1}{2}\mathbf{q}\cdot\mathbf{r})$ in the form factor integrals and the transitions to higher angular momentum states. Castillejo and Singh[7] have made calculations similar to those in this paper, but they also neglected higher angular momentum states. It is essential to include these states in order to get detailed agreement with the experimental data. The theory of slightly-inelastic electron-deuteron scattering, which in many respects is similar to inelastic proton-deuteron scattering, has been investigated by Durand.[8] His analysis is of experiments involving large momentum transfer (1.8 to 2.8 F^{-1}) and its primary purpose is to study the detailed structure of the low-energy final-state wave functions. The purpose of the present analysis is to study the high-energy proton-neutron interaction.

The slightly-inelastic proton-deuteron cross section is found to depend on $(\sigma_{np}+\sigma_{pp})$, the sum of the free proton-neutron and proton-proton cross sections, and on two other parameters, Σ_s and Σ_t, which are also functions of the nucleon-nucleon scattering amplitudes. These parameters are defined in Eqs. (3.9) and (3.10). In impulse approximation Σ_t is also related to elastic proton-deuteron scattering. However slightly-inelastic scattering mainly depends on Σ_s. In Sec. IV the theory is used to analyze the measurements of Stairs, Wilson, and Cooper[9] at 158 MeV. The parameters $(\sigma_{np}+\sigma_{pp})$, Σ_t, and Σ_s are found by making a least-squares adjustment to the data. The experimental results in this way are interpreted as yielding a measurement of Σ_s at the laboratory angles 5°, 10°, 15°, and 20°. The values obtained are found to differ by as much as 40% from the values predicted by the phase-shift solutions of Breit *et al.*[10] and others.

II. THEORY OF INELASTIC PROTON-DEUTERON SCATTERING

In this section we derive the differential cross section $d\sigma/d\Omega_p dE'$ for the process in which a high-energy incident proton is scattered from a deuteron into the solid angle $d\Omega_p$ and transfers enough energy to disintegrate the deuteron, coming out with a final energy E'. Let $\mathbf{P}$ and $\mathbf{P}'$ be the initial and final momenta of the incident proton and let $\mathbf{k}_p$, $\mathbf{k}_p'$ and $\mathbf{k}_n$, $\mathbf{k}_n'$ be the initial and final momenta of the target proton and neutron, respectively. Since initially the target nucleons are bound to form a deuteron at rest, we have $\mathbf{k}_n=-\mathbf{k}_p$. Then, from momentum conservation the momentum transfer is

$$\mathbf{q}=\mathbf{P}-\mathbf{P}'=\mathbf{k}_n'+\mathbf{k}_p'. \tag{2.1}$$

The relative momentum in the center of mass of the resultant neutron-proton system is

$$\mathbf{k}=\tfrac{1}{2}(\mathbf{k}_n'-\mathbf{k}_p'). \tag{2.2}$$

Experimentally the magnitude and direction of $\mathbf{P}'$ are measured. Knowing $\mathbf{P}$ this determines $\mathbf{q}$ and the magnitude of $\mathbf{k}$. The latter follows from energy conservation. Assuming that $k_p'^2$, $k_n'^2$ and $(P^2-P'^2)$ are small compared to m^2, a semirelativistic expression for the difference between the final and initial energies is

$$E_f-E_i=\tfrac{1}{2}(P'^2-P^2)/E+(\tfrac{1}{4}q^2+k^2)/m+\epsilon_B=0. \tag{2.3}$$

Here $E=(P^2+m^2)^{1/2}$ is the total energy of the incident proton, m is the mass of the proton, and ϵ_B is the absolute magnitude of the binding energy of the deuteron.

The momenta $\mathbf{k}_p'$ and $\mathbf{k}_n'$ are not determined individually. They are given in terms of q and k by

$$\mathbf{k}_n'=\tfrac{1}{2}\mathbf{q}+\mathbf{k}, \tag{2.4}$$

$$\mathbf{k}_p'=\tfrac{1}{2}\mathbf{q}-\mathbf{k}, \tag{2.5}$$

and so depend on the angle between $\mathbf{q}$ and $\mathbf{k}$. In the special case when $\mathbf{k}=\frac{1}{2}\mathbf{q}$ we have $\mathbf{k}_p'=0$ and $\mathbf{k}_n'=\mathbf{q}$; the neutron carries off all the momentum transferred to the deuteron and the proton is left at rest. In the case $\mathbf{k}=-\frac{1}{2}\mathbf{q}$ the situation is reversed. These are just the conditions for quasi-free proton-neutron and proton-proton scattering, respectively. A broad peak is expected in the differential cross section for values of E' corresponding to $k=\frac{1}{2}q$. For scattering angles $\lesssim 20°$ this peak is largely masked, as we shall see, by a peak due to the strong final-state interaction between the resultant

[6] R. Gluckstern and H. Bethe, Phys. Rev. **81**, 761 (1951).

[7] L. Castillejo and L. Singh, in *Nuclear Forces and the Few-Nucleon Problem*, edited by T. C. Griffith and E. A. Power (Pergamon Press, New York, 1960).

[8] L. Durand, III, Phys. Rev. **123**, 1393 (1961).

[9] D. Stairs, R. Wilson, and P. Cooper, Jr., preceding paper [Phys. Rev. **128**, 1672 (1963)]. This paper will be referred to as SWC.

[10] G. Breit, M. Hull, K. Lassila, and K. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960); M. Hull, K. Lassila, H. Ruppel, F. McDonald, and G. Breit, *ibid.* **122**, 1606 (1961).

target nucleons. However, for larger scattering angles quasi-free processes play an important role. We shall discuss them in more detail in a later paper.

We wish to relate the proton-deuteron scattering matrix to the free nucleon-nucleon scattering matrix. Chew and Goldberger[11] have shown in general that if the T matrix for the total Hamiltonian is expanded in terms of the two-body t matrices one gets $T=t_n+t_p$ plus terms which represent the multiple scattering of the incident proton from the two target nucleons and the effect of the binding of the target nucleons. Here t_n and t_p are the t matrices for free proton-neutron and proton-proton scattering, respectively. In the spirit of the impulse approximation we neglect the additional terms in the expansion of T and so obtain the following for the transition matrix between two states Φ_a and Φ_b of the unperturbed system:

$$T_{ba}=(\Phi_b^{(-)},t_n\Phi_a)+(\Phi_b^{(-)},t_p\Phi_a). \quad (2.6)$$

In the present problem the initial state is

$$\Phi_a=\Phi_0=(2\pi)^{-3}\exp[i\tfrac{1}{2}\mathbf{K}\cdot(\mathbf{r}_n+\mathbf{r}_p)]\times\exp(i\mathbf{P}\cdot\mathbf{r}_0)\phi_0(\mathbf{r}_n-\mathbf{r}_p), \quad (2.7)$$

where ϕ_0 is the deuteron ground-state wave function. Here $\mathbf{r}_0$, $\mathbf{r}_p$, and $\mathbf{r}_n$ are the position vectors of the incident proton, the target proton, and the target neutron, respectively. $\mathbf{K}$ is the initial momentum of the center of mass of the deuteron (which is presumably zero). The final-state wave functions is

$$\Phi_b^{(-)}=\Phi_\mathbf{k}^{(-)}=(2\pi)^{-3}\exp[i\tfrac{1}{2}\mathbf{K}'\cdot(\mathbf{r}_n+\mathbf{r}_p)]\times\exp(i\mathbf{P}'\cdot\mathbf{r}_0)\phi_\mathbf{k}^{(-)}(\mathbf{r}_n-\mathbf{r}_p), \quad (2.8)$$

where $\phi_\mathbf{k}^{(-)}$ is a neutron-proton incoming scattering state of relative momentum $\mathbf{k}$. Using these wave functions we get

$$T_n=(\Phi_\mathbf{k}^{(-)},t_n\Phi_0)=\delta^3(\mathbf{K}+\mathbf{P}-\mathbf{K}'-\mathbf{P}')t_n(q)F(\mathbf{q},\mathbf{k}). \quad (2.9)$$

Here

$$F(\mathbf{q},\mathbf{k})=\int[\phi_\mathbf{k}^{(-)}(\mathbf{r})]^*\phi_0(\mathbf{r})\exp(i\tfrac{1}{2}\mathbf{q}\cdot\mathbf{r})d\mathbf{r}, \quad (2.10)$$

and

$$t_n(q)\delta^3(\mathbf{k}_n+\mathbf{P}-\mathbf{k}_n'-\mathbf{P}')=\langle\mathbf{k}_n',\mathbf{P}'|t_n|\mathbf{k}_n,\mathbf{P}\rangle. \quad (2.11)$$

That is, $t_n(q)$ is the nonsingular part of the t matrix.

In general the states $|\mathbf{k},\mathbf{P}\rangle$ and $|\mathbf{k}_n',\mathbf{P}'\rangle$ do not have the same energy, so that the t matrix in (2.11) does not resemble the t matrix for free proton-neutron scattering. Using Eq. (2.3) it is easily shown that ΔE, the energy difference between these two states, is

$$\Delta E=-\epsilon_B-(\mathbf{k}-\tfrac{1}{2}\mathbf{q})^2/m. \quad (2.12)$$

Thus ΔE will be small under two conditions. Either both $\mathbf{k}$ and $\mathbf{q}$ are small, which is the condition for slightly-inelastic scattering, or else $\mathbf{k}=\frac{1}{2}\mathbf{q}$, which is the condition for quasi-free scattering. In these two special cases it is possible to use the free t matrix to describe $t_n(q)$.

We get T_p by interchanging the labels "p" and "n" everywhere in Eq. (2.9). Recalling Eq. (2.2) we get

$$T_p=\delta^3(\mathbf{K}+\mathbf{P}-\mathbf{K}'-\mathbf{P}')t_p(q)F(\mathbf{q},-\mathbf{k})$$

and so

$$T=\delta^3(\mathbf{K}+\mathbf{P}-\mathbf{K}'-\mathbf{P}')\times[t_n(q)F(\mathbf{q},\mathbf{k})+t_p(q)F(\mathbf{q},-\mathbf{k})]. \quad (2.13)$$

The t matrices are related to the nucleon-nucleon center-of-mass scattering matrix M by

$$t(q)=-(2\pi/m)(P/P_0)M(q), \quad (2.14)$$

where P and P_0 are the laboratory and nucleon-nucleon center-of-mass momenta, respectively. Since the deuteron is initially in a triplet spin state we must multiply Eq. (2.13) on the right by Λ_t, the triplet projection operator, obtaining

$$T=-\delta^3(\mathbf{P}+\mathbf{K}-\mathbf{P}'-\mathbf{K}')(2\pi/m)(P/P_0)\mathfrak{M}, \quad (2.15)$$

where

$$\mathfrak{M}=[M_{np}(q)F(\mathbf{q},\mathbf{k})+M_{pp}(q)F(\mathbf{q},-\mathbf{k})]\Lambda_t. \quad (2.16)$$

Here M_{np} and M_{pp} are the proton-neutron and proton-proton scattering matrices.

The cross section then is

$$d\sigma=(m/p)(P/mP_0)^2\delta(E_f-E_i)\tfrac{1}{6}\mathrm{Tr}(\mathfrak{M}^\dagger\mathfrak{M})d\mathbf{P}'d\mathbf{k}. \quad (2.17)$$

The trace averages over initial and sums over final spin states. We write $d\mathbf{k}=k^2d\Omega_kdk$ and integrate with respect to dk to remove the delta function. Using (2.3) we get

$$d\sigma/d\Omega_pdE'=(d\sigma/d\Omega_pdP')(dP'/dE')=\tfrac{1}{6}D\,\mathrm{Tr}(\mathfrak{M}^\dagger\mathfrak{M})d\Omega_k, \quad (2.18)$$

where

$$D=PP'k/(2mP_0^2) \quad (2.19)$$

is the kinematical factor. Finally we must integrate with respect to $d\Omega_k$ since only the magnitude of $\mathbf{k}$ is determined. Thus

$$d\sigma/d\Omega_pdE'=\tfrac{1}{6}D\int\mathrm{Tr}(\mathfrak{M}^\dagger\mathfrak{M})d\Omega_k. \quad (2.20)$$

Since the neutron-proton potential is spin-dependent, the final-state wave function $\phi_\mathbf{k}$ and hence the form factor $F(\mathbf{q},\mathbf{k})$ will be spin dependent. It is necessary, therefore, to write the scattering matrix as the sum of two parts: one part describing transitions to a triplet final state and the other transitions to a singlet final state. Using Λ_t and Λ_s, the triplet and singlet projection

[11] G. Chew and M. Goldberger, Phys. Rev. 98, 778 (1952).

operators, we can write

$$\mathfrak{M}=\Lambda_t[M_{np}(q)\mathfrak{F}_t(\mathbf{q},\mathbf{k})+M_{pp}(q)\mathfrak{F}_t(\mathbf{q},-\mathbf{k})]\Lambda_t+\Lambda_s[M_{np}(q)\mathfrak{F}_s(\mathbf{q},\mathbf{k})+M_{pp}(q)\mathfrak{F}_s(\mathbf{q},-\mathbf{k})]\Lambda_t, \quad (2.21)$$

where $\mathfrak{F}_s$ and $\mathfrak{F}_t$ are given by Eq. (2.10) with the appropriate singlet and triplet wave functions, respectively, inserted for $\phi_{\mathbf{k}}^{(-)}$.

Furthermore, if k, the relative momentum in the center of mass of the resultant neutron-proton system, is small, then only the final-state interaction in the S state need be considered. The total wave function in the singlet state can then be written

$${}^s\phi_{\mathbf{k}}={}^s\chi_{\mathbf{k}}+\psi_{\mathbf{k}}, \quad (2.22)$$

where ${}^s\chi_{\mathbf{k}}$ is the singlet S wave function and

$$\psi_{\mathbf{k}}=(2\pi)^{-3/2}[e^{i\mathbf{k}\cdot\mathbf{r}}-\sin kr/kr] \quad (2.23)$$

is the plane wave function for all higher angular momentum states. Similarly, the triplet wave function is

$${}^t\phi_{\mathbf{k}}={}^t\chi_{\mathbf{k}}+\psi_{\mathbf{k}}. \quad (2.24)$$

The form factors can then be written

$$\mathfrak{F}_s=F_s+F_B-F_0, \quad (2.25)$$

$$\mathfrak{F}_t=F_t+F_B-F_0. \quad (2.26)$$

Here F_s and F_t are the form factors of Eq. (2.10) evaluated with ${}^s\chi_{\mathbf{k}}^{(-)}$ and ${}^t\chi_{\mathbf{k}}^{(-)}$, respectively, in place of $\phi_{\mathbf{k}}^{(-)}$, and F_B and F_0 are the Born approximation form factors evaluated with $(2\pi)^{-3/2}e^{i\mathbf{k}\cdot\mathbf{r}}$ and $(2\pi)^{-3/2}\times\sin kr/kr$, respectively.

For the S-wave form factors we have

$$F_0(\mathbf{q},\mathbf{k})=F_0(\mathbf{q},-\mathbf{k}) \quad (2.27)$$

and similarly for F_s and F_t. Then the scattering matrix can be written

$$\begin{aligned}\mathfrak{M}=\Lambda_t(M_{np}+M_{pp})F_t(\mathbf{q},\mathbf{k})\Lambda_t &+\Lambda_s(M_{np}+M_{pp})F_s(\mathbf{q},\mathbf{k})\Lambda_t\\ &+\{M_{np}[F_B(\mathbf{q},\mathbf{k})-F_0(\mathbf{q},\mathbf{k})]\\ &+M_{pp}[F_B(\mathbf{q},-\mathbf{k})-F_0(\mathbf{q},\mathbf{k})]\}\Lambda_t. \quad (2.28)\end{aligned}$$

In the next section we shall use these results to calculate the cross section given by Eq. (2.20).

III. CALCULATION OF THE CROSS SECTION

The cross section is calculated by inserting the expression for $\mathfrak{M}$ given by Eq. (2.28) into Eq. (2.20). Since (F_B-F_0) represents that part of the final-state wave function with angular momentum greater than zero, the last term in Eq. (2.28) is orthogonal to the first two terms upon integrating with respect to $d\Omega_k$. Furthermore, since $\Lambda_s\Lambda_t=0$ we see that the first two terms give no cross term. Finally we note that F_t, F_s, and F_0 are independent of the direction of $\mathbf{k}$ and so integrating these terms simply yields 4π. We get then for the cross section:

$$\begin{aligned}d\sigma/d\Omega_p dE' =(4\pi D/6)\{&\mathrm{Tr}[(M_{np}^\dagger+M_{pp}^\dagger)\Lambda_t(M_{np}+M_{pp})\Lambda_t]\\ &\times|F_t|^2+\mathrm{Tr}[(M_{np}^\dagger+M_{pp}^\dagger)\Lambda_s\\ &\times(M_{np}+M_{pp})\Lambda_t]|F_s|^2\\ &+\mathrm{Tr}[(M_{np}^\dagger M_{np}+M_{pp}^\dagger M_{pp})\Lambda_t]\\ &\times[M(k,q)-|F_0|^2]\\ &+\mathrm{Tr}[2\,\mathrm{Re}(M_{np}^\dagger M_{pp})\Lambda_t]\\ &\times[N(k,q)-|F_0|^2]\}. \quad (3.1)\end{aligned}$$

Here

$$M(k,q)=(4\pi)^{-1}\int[F_B(\mathbf{q},\mathbf{k})]^2d\Omega_k, \quad (3.2)$$

$$N(k,q)=(4\pi)^{-1}\int F_B(\mathbf{q},\mathbf{k})F_B(\mathbf{q},-\mathbf{k})d\Omega_k. \quad (3.3)$$

The cross section thus splits into four terms: one term representing the contribution of the triplet S final state, one the contribution of the singlet S state, and two the contribution of all higher angular momentum states. We proceed to calculate these terms.

A. S-State Terms

In the case of slightly-inelastic proton-deuteron scattering the relative momentum k in the center of mass of the resultant neutron-proton system is small, so that the main contribution to the cross section will come from events which leave the final-state neutron-proton system in a relative S state. This contribution is given by the first two terms in (3.1). The cross sections for triplet and singlet S-state scattering are

$$(d\sigma/d\Omega_p dE')_t=4\pi D|F_t(\mathbf{q},\mathbf{k})|^2\Sigma_t, \quad (3.4)$$

$$(d\sigma/d\Omega_p dE')_s=4\pi D|F_s(\mathbf{q},\mathbf{k})|^2\tfrac{1}{3}\Sigma_s, \quad (3.5)$$

where

$$\Sigma_t=\tfrac{1}{6}\,\mathrm{Tr}[(M_{np}^\dagger+M_{pp}^\dagger)\Lambda_t(M_{np}+M_{pp})\Lambda_t], \quad (3.6)$$

$$\tfrac{1}{3}\Sigma_s=\tfrac{1}{6}\,\mathrm{Tr}[(M_{np}^\dagger+M_{pp}^\dagger)\Lambda_s(M_{np}+M_{pp})\Lambda_t]. \quad (3.7)$$

Using the notation of Kerman, McManus and Thaler[2] the nucleon-nucleon scattering matrix can be written

$$\begin{aligned}M_T=A_T+B_T(\boldsymbol{\sigma}_1\cdot\mathbf{n})(\boldsymbol{\sigma}_2\cdot\mathbf{n})+C_T(\boldsymbol{\sigma}_1\cdot\mathbf{n}+\boldsymbol{\sigma}_2\cdot\mathbf{n})\\ +E_T(\boldsymbol{\sigma}_1\cdot\mathbf{q})(\boldsymbol{\sigma}_2\cdot\mathbf{q})+F_T(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p}). \quad (3.8)\end{aligned}$$

Here T, which is either 0 or 1, labels the two possible isotopic spin states of the nucleon-nucleon system. The subscripts 1 and 2 label the spin operators for the two interacting nucleons, and $\mathbf{n}$, $\mathbf{p}$, $\mathbf{q}$ are unit vectors in the directions $\mathbf{P}_0\times\mathbf{P}_0'$, $\mathbf{P}_0+\mathbf{P}_0'$, and $\mathbf{P}_0'-\mathbf{P}_0$, respectively. The proton-proton system is a pure isotopic spin 1 state and the neutron-proton system is an equal mixture of

$T=0$ and $T=1$, so

$$M_{pp}=M_1,$$
$$M_{np}=(M_1+M_0)/2.$$

Then evaluating the traces in Eqs. (3.4) and (3.5) we find:

$$\Sigma_t=\tfrac{1}{4}\{|3A_1+A_0|^2+(5/3)|3C_1+C_0|^2 +\tfrac{2}{3}(|3B_1+B_0|^2+|3E_1+E_0|^2+|3F_1+F_0|^2)\}, \quad (3.9)$$

$$\Sigma_s=\tfrac{1}{4}\{|C_1-C_0|^2+|B_1-B_0|^2 +|E_1-E_0|^2+|F_1-F_0|^2\}. \quad (3.10)$$

There are a number of important features to be noted about these results. In the first place we see that the triplet cross section depends on the nucleon-nucleon amplitudes in the same way as does the elastic proton-deuteron cross section,[2] whereas the singlet cross section has quite a different dependence. In fact, as already noted, in the singlet cross section the $T=0$ and $T=1$ amplitudes occur with the same statistical weight factor. Secondly, since the deuteron ground-state wave function is orthogonal to the triplet scattering states, we see from Eq. (2.10) that $F_t(0,k)=0$. Hence for small q the singlet cross section will be much larger than the triplet cross section, in spite of the statistical weight factor of $\frac{1}{3}$ multiplying the singlet cross section. Thirdly, since the Coulomb interaction between the two protons is largely spin-independent, it occurs mostly as a modification of A_1. Thus the Coulomb interaction does not enter into Σ_s.

In evaluating $F_s(\mathbf{q},\mathbf{k})$ and $F_t(\mathbf{q},\mathbf{k})$ it is important to use exact scattering state wave functions. To do this we have used square well potentials to describe the low-energy neutron-proton potential. The well parameters were chosen to fit the effective range parameters. The well parameters used for the singlet and triplet potentials are given in Table I, together with the values of the scattering length and effective range calculated from these potentials. The tensor force has been neglected.

The calculation should not be sensitive to the details of the potential (or equivalently, to the structure of the wave functions at small distances) because it is limited to momentum transfer of at most 1.0 F^{-1}. Electron-deuteron cross-section measurements, both elastic[12] and slightly-inelastic,[13] do show evidence of a hard core in the neutron-proton potential. However these effects only appear at momentum transfers of 1.6 F^{-1} or more. For $q\leq 1.0$ F^{-1} all reasonable deuteron wave functions give essentially the same deuteron form factor. Furthermore, Durand's calculation[8] of the slightly-inelastic electron-deuteron cross section only shows the effect of a hard core in the neutron-proton potential at a momentum transfer greater than 1.5 F^{-1}. It is interesting to note that in at least one case[14] the square well potential and a repulsive core potential both give the same deuteron form factor for momentum transfer up to 2.4 F^{-1}.

TABLE I. Square-well parameters used for the singlet and triplet state together with the scattering length, a, and the effective range, r_0, calculated from these potentials.

State	Well depth (MeV)	Well width (F)	Scattering length, a (F)	Effective range, b (F)
Triplet	34.5	2.074	5.39	1.75
Singlet	14.54	2.545	−23.74	2.65

[12] J. Friedman, H. Kendall, and P. Gram, Phys. Rev. **120**, 992 (1960); J. McIntyre and G. Burleson, Phys. Rev. **112**, 2077 (1958).

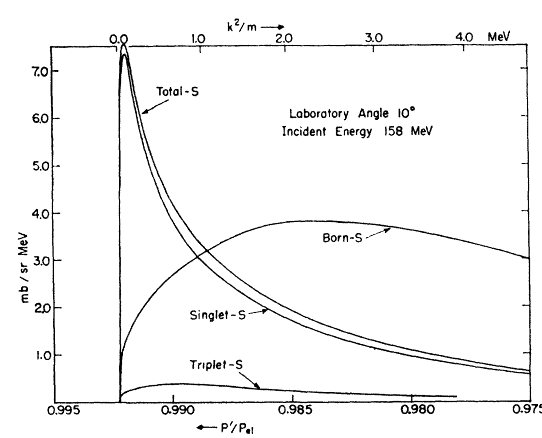

FIG. 1. The S-wave contribution to the inelastic proton-deuteron cross section at 10° for 158-MeV incident protons. The figure shows the singlet-S and the triplet-S contributions, their sum, and also the S-wave contribution neglecting final-state interaction (Born-S).

The normalized deuteron wave function calculated from the triplet potential was used, together with the scattered wave functions ${}^t\chi_\mathbf{k}$ and ${}^s\chi_\mathbf{k}$, calculated from these potentials, to calculate the functions $F_s(\mathbf{q},\mathbf{k})$ and $F_t(\mathbf{q},\mathbf{k})$. The results for 158-MeV incident protons scattered through 10° are shown in Fig. 1. Here are plotted the singlet and triplet cross sections separately, their sum, and also the S-wave "Born approximation" cross section (that is, the S-wave cross section calculated neglecting final-state interaction). This latter quantity is given by the sum of Eqs. (3.4) and (3.5) with F_t and F_s replaced by F_0. The values of Σ_s and Σ_t used are given in Table II. The cross sections are in millibarns/steradian MeV. Along the abscissa we have plotted P'/P_{el} where P_{el} is the momentum of an elastically scattered proton; also along the abscissa, the energy in the center of mass of the resultant neutron-proton system, k^2/m, is shown.

As expected the triplet cross section is nearly zero. The singlet cross section is sharply peaked at the low-

[13] H. Kendall, J. Friedman, E. Erickson, and P. Gram, Phys. Rev. **124**, 1596 (1961).

[14] J. McIntyre, Phys. Rev. **103**, 1464 (1956). The remarkable similarity between the form factors for a hard core and square well potential up to $q\approx 2.4$ F^{-1} suggests that in other calculations (such as those of references 8 and 13) the square well might be indistinguishable from a hard core potential for $q\lesssim 2.0$ F^{-1}.

energy[15] end of the spectrum, falling to one half of its peak value within 1 MeV of threshold. This is due to the strong final-state interaction between the target neutron and proton. When this interaction is neglected the whole character of the spectrum is changed. A broad maximum occurs at about 2.25 MeV above threshold, which is just the binding energy of the deuteron. The Born-approximation cross section in this region is much larger than the exact cross sections, since in Born approximation the triplet-state contribution, instead of being small, is 3 times the singlet contribution, the factor 3 being just the statistical weight of the triplet state.

At smaller angles the singlet peak increases; it is 13.6 mb/sr MeV at 5°. The triplet cross section decreases at smaller angles, being less than 1% of the singlet cross section at 5°. At larger angles the singlet peak decreases and the triplet cross section increases. At 15° the singlet peak is only 3.4 mb/sr MeV; the maximum of the triplet cross section is 10% of the singlet peak, but occurs at a somewhat higher energy. The effect of this is to lower and broaden the spectrum with increasing angle.

B. Higher Angular Momentum States

The last two terms of Eq. (3.1) give the contribution to the cross section of all states with angular momentum greater than zero. This contribution is expected to increase with k; as we shall see, it also increases with q, and hence with the scattering angle θ.

The form factors F_B and F_0 are evaluated from Eq. (2.10) using $(2\pi)^{-3/2}e^{i\mathbf{k}\cdot\mathbf{r}}$ and $(2\pi)^{-3/2}\sin kr/kr$, respectively, for $\phi_\mathbf{k}$. For ϕ_0, the deuteron ground-state wave function, we use the simple Hulthén wave function given by Moraviscik[16]:

$$\phi_0=0.26[\exp(-\alpha r)-\exp(-\beta r)]/r,\quad \alpha=0.232\ \mathrm{F}^{-1},\quad \beta=1.202\ \mathrm{F}^{-1}. \tag{3.11}$$

This enables us to perform the integrals with respect to $d\Omega_k$ indicated in Eqs. (3.2) and (3.3). The results appear in Durand's article[8] as Eqs. (29) and (30). The use of a different deuteron wave function to treat the higher angular momentum states is justified again because the calculation is limited to small momentum transfer. The traces in the last two terms of Eq. (3.1) are

$$\tfrac{1}{6}\,\mathrm{Tr}[(M_{np}{}^\dagger M_{np}+M_{pp}{}^\dagger M_{pp})\Lambda_t]=\sigma_{np}+\sigma_{pp}, \tag{3.12}$$

$$\tfrac{1}{6}\,\mathrm{Tr}[2\,\mathrm{Re}(M_{np}{}^\dagger M_{pp})\Lambda_t]=\Sigma_t+\tfrac{1}{3}\Sigma_s-\sigma_{np}-\sigma_{pp}. \tag{3.13}$$

Here σ_{np} and σ_{pp} are the free neutron-proton and proton-proton differential cross sections in the nucleon-nucleon center-of-mass system and Σ_t and Σ_s are given by Eqs. (3.9) and (3.10). The trace in Eq. (3.13) is the interference term between the scattering of the incident proton from the target proton and from the target neutron. From the last two terms of Eq. (3.1), we get for the contribution to the cross section from higher angular momentum states:

$$(d\sigma/d\Omega_p dE')_{L>0}=4\pi D\{(\sigma_{np}+\sigma_{pp})M(k,q)+[\Sigma_t+\tfrac{1}{3}\Sigma_s-\sigma_{np}-\sigma_{pp}]N(k,q)-[\Sigma_t+\tfrac{1}{3}\Sigma_s]F_0{}^2\}. \tag{3.14}$$

The total slightly-inelastic cross section is given by the sum of Eqs. (3.4), (3.5), and (3.14).

In Eq. (3.14) the first term comes from the direct scattering of the incident nucleon by the target nucleons. The second term arises from interference of these two events. The third term is the S-wave contribution in Born approximation which must be subtracted out.

From Eq. (3.14) we see that the ratio N/M is a measure of the importance of the interference term relative to the direct cross-section term. From the explicit form of M and N[8], it is found that their ratio depends on the parameter

$$X=(\mathbf{k}\cdot\mathbf{q}-m\Delta E)/kq, \tag{3.15}$$

where ΔE is given by (2.12). In the slightly-inelastic region, $kq\to 0$ and $X\to\infty$. In this limit $N/M\to 1$ and there is complete interference. In the quasi-free region, $\mathbf{k}=\frac{1}{2}\mathbf{q}$ and so $X\approx 1$. In this limit $N/M\to 0$ and the direct cross-section term dominates. In quasi-free scattering the inelastic cross section is just proportional to the free nucleon-nucleon cross sections.

It is a fundamental principle of quantum mechanics that the probability amplitudes for two processes interfere whenever these processes are indistinguishable. Now it may at first be thought that it is possible to determine whether the incident proton scatters from a

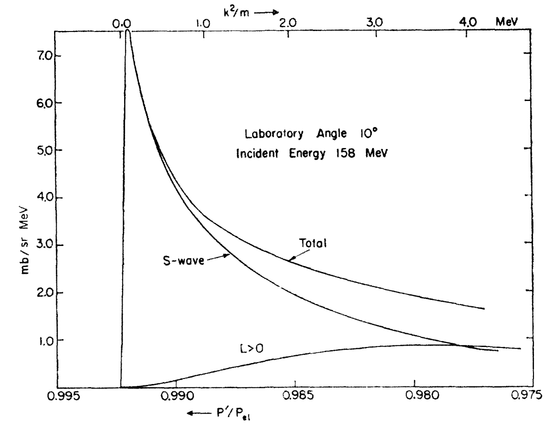

FIG. 2. The inelastic proton-deuteron cross section at 10° for 158-MeV incident protons. The figures show the total S-wave contribution, the contribution from higher angular momentum states ($L>0$), and their sum.

[15] When referring to the abscissa of the figures in this and the accompanying article (SWC) we shall always mean the energy in the center of mass of the resultant neutron-proton system. This quantity increases to the right in the figures whereas the momentum and energy of the scattered proton decrease to the right.

[16] M. Moraviscik, Nuclear Phys. 7, 113 (1958).

target neutron or a target proton by observing which target particle carries off the recoil momentum. However, because of the internal momentum of the deuteron, the recoil particle may not be the particle with which the incident proton interacted. For example, the incident proton could interact with a target proton with

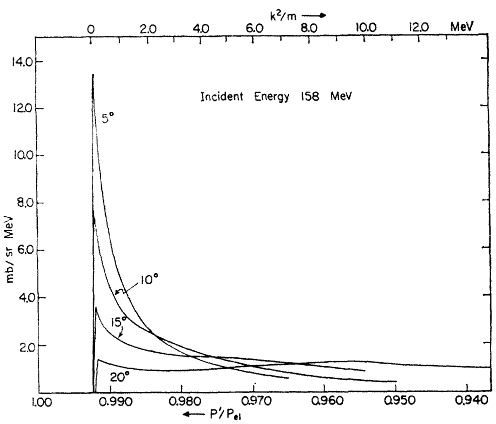

FIG. 3. The inelastic proton-deuteron cross section at 5°, 10°, 15°, and 20° for 158-MeV incident protons, calculated using the parameters in Table II.

internal momentum $\mathbf{k}_p=-\mathbf{q}$; the target neutron has $\mathbf{k}_n=\mathbf{q}$. Then the incident proton transfers momentum $\mathbf{q}$ to the target proton, bringing it to rest in the laboratory system, and the target neutron carries off the recoil momentum $\mathbf{q}$. The probability of finding a target proton with internal momentum q decreases rapidly for $q>(m\epsilon_B)^{1/2}$, so for large q the recoil particle will in fact be the particle with which the incident proton interacted, and so the interference term disappears; slightly-inelastic scattering changes continuously into quasi-free scattering. We shall discuss quasi-free scattering in a subsequent article.

Figure 2 gives the results at 10°. The exact S-wave contribution is plotted together with $(d\sigma/d\Omega_p dE')_{L>0}$ and their sum. The contribution of the higher angular momentum states peaks at a smaller value of P' (larger value of k) than does the S-wave contribution. Hence the main effect of this contribution is to broaden out the peak in the spectrum. Also, since the S-wave cross section falls off rapidly at smaller P', the higher angular momentum states give the dominant contribution to the tail of the spectrum.

Figure 3 compares the total cross sections at 5°, 10°, 15°, and 20°, again using the parameters in Table II. The spectrum at 5° is due almost entirely to the singlet S state. At 10° the singlet S state still dominates, but the other terms in the cross section somewhat modify the spectrum. At 15° the singlet-S contributes still less to the cross section. While it accounts for nearly 97% of the cross section right at the peak, it falls off rapidly while the other terms increase with decreasing P'; it accounts for only 50% of the cross section at $P'/P_{\rm el}=0.9800$. At 20° two distinct peaks appear, the slightly-inelastic and the quasi-free peaks.

In the next section the theory will be used to analyze the experimental measurements of Stairs *et al.*[9]

IV. ANALYSIS OF THE DATA

Stairs, Wilson, and Cooper[9] have measured the proton-deuteron inelastic cross section at 158 MeV using a high-resolution magnetic spectrometer. The measurements were made at 5°, 10°, 15°, and 20° for values of E' ranging from threshold to about 10 MeV above threshold. The results are shown in Figs. 5, 6, 7, and 8 of SWC.

Combining Eqs. (3.4), (3.5), and (3.14) we find that the cross section can be written

$$d\sigma/d\Omega_p dE'=\tfrac{1}{3}\alpha\Sigma_s+\beta\Sigma_t+\gamma(\sigma_{np}+\sigma_{pp}), \qquad (4.1)$$

where

$$\alpha=4\pi D(F_s^2+N-F_0^2), \qquad (4.2)$$

$$\beta=4\pi D(F_t^2+N-F_0^2), \qquad (4.3)$$

$$\gamma=4\pi D(M-N). \qquad (4.4)$$

The coefficients α, β, and γ are functions of q and k but do not depend upon the nucleon-nucleon scattering matrix. The parameters Σ_s, Σ_t, σ_{np}, and σ_{pp} are assumed to depend only on q, which, for a given scattering angle, is nearly constant over the range of k under consideration. These parameters are functions of the nucleon-nucleon scattering amplitudes. Assuming these parameters are constant at a given angle, the data at any angle can be fitted by adjusting only three parameters, Σ_s, Σ_t, and $(\sigma_{pp}+\sigma_{np})$.

To analyze the data, the coefficients α, β, and γ were calculated at each angle as functions of $P'/P_{\rm el}$. The experimental momentum resolution curve was then folded into these coefficients to correct them for finite resolution. Using the corrected coefficients in Eq. (4.1) a least-squares fit was made to the data at each angle. The best fits obtained are shown in Figs. 5, 6, 7, and 8 of SWC and the values of the parameters found are given in Table II. Table II also lists the number of data fitted at each angle and the value of χ^2 obtained.

At 5° the singlet S state so dominates that effectively only Σ_s is determined. The fit is shown in Fig. 5 of SWC.

TABLE II. The values of the parameters Σ_s, Σ_t, and $(\sigma_{np}+\sigma_{pp})$ obtained from a least-squares analysis of the 158-MeV slightly inelastic proton-deuteron scattering data of Stairs *et al.*[a] The number of points fitted and the value of χ^2 obtained are also given.

Laboratory angle	Σ_s (mb)	Σ_t (mb)	$\sigma_{np}+\sigma_{pp}$ (mb)	No. of points fitted	χ^2
5°	12.17±0.25	−30.7 ±48.2	0.2 ±21.6	9	9.8
5°	(12.01±0.12)	(7.17)	(17.1)	9	10.5
10°	11.50±0.70	10.0 ± 6.9	10.8 ± 2.5	15	14.0
15°	10.32±0.54	7.2 ± 1.8	8.71± 0.44	10	15.9
20°	6.70±0.45	8.09± 0.74	9.17± 0.15	8	14.2

[a] See reference 9.

The indeterminateness of Σ_t and $(\sigma_{np}+\sigma_{pp})$ is even greater than indicated by the large errors, since these errors have, in addition, a large positive correlation. If the values of Σ_t and $(\sigma_{np}+\sigma_{pp})$ are constrained to have values near those indicated by other considerations (see Tables III and IV), χ^2 increased by only 7%. The value of Σ_t and $(\sigma_{np}+\sigma_{pp})$ used and the resulting value of Σ_s are given in parentheses in Table II. The actual points fitted are plotted as black circles. A χ^2 of 9.8 was obtained for a fit to 9 points. The two points at the lowest energy were not fitted since they actually occur below threshold and so the fit is very sensitive to the exact manner in which the resolution is folded in. The two points at the higher energy could not be fitted adequately and so were not included in the final determination of Σ_s. No good reason is known why these two points cannot be fitted.

The situation is better at 10°. Here all but the very first point was fitted, a total of 14 points, with a resulting χ^2 of 14.0. The fit is shown in Fig. 6 of SWC. As is expected the errors on Σ_t and $(\sigma_{np}+\sigma_{pp})$ are smaller now, since the triplet-S and higher angular momentum states are more important than at 5°. However, there is still a large positive correlation between these parameters.

The 15° results are shown in Fig. 7 of SWC. An indication of the quasi-free peak is seen in the broadening of the slightly-inelastic peak. In fitting these data the first point and the last three points were omitted. The selection of the points to be fitted is somewhat arbitrary. However, no other choice of points gave a reasonable fit. It is hard to understand why the last points cannot be fitted better than they are.

Figure 8 of SWC shows that two peaks are clearly resolved at 20°. The singlet-S peak is now just a bump on the lower energy end of the broad quasi-free peak. In the least-square fit, two points near threshold were excluded because they are not consistent with neighboring points. The two points at the high-energy end were also excluded because they could under no circumstance be fitted and their inclusion distorted the rest of the fit. The complete disagreement here at high energy is even more serious than at 15°. At energies where these discrepancies occur the effective range theory used to calculate F_t and F_s is certainly not valid. However, the values of F_t and F_s in this region are so small that even large percentage errors in them could not affect the cross section appreciably, the important contributions coming from the higher angular momentum states. The most reasonable guess, apart from experimental error, is that the discrepancy is due to effects arising from final-state interaction in P or D states.

TABLE III. Comparison of the values of Σ_t obtained from the analysis of inelastic proton-deuteron scattering, from elastic proton-deuteron scattering, and from the phase-shift solutions of Breit *et al.*,[a] Prenowitz,[b] and Kerman, McManus, and Thaler.[c]

Laboratory angle	Inelastic p-d (mb)	Elastic p-d (mb)	YLAM YLAN3M (mb)	Prenowitz YLAN3M (mb)	KMT (mb)
5°		7.82±0.35	6.92	7.18	6.82
10°	10.0 ±6.9	11.52±0.70	10.89	11.49	11.54
15°	7.2 ±1.8	9.79±0.37	11.32	11.94	11.59
20°	8.09±0.74	8.59±0.80	10.11	11.02	9.76

[a] See reference 10.
[b] See reference 18.
[c] See reference 2.

TABLE IV. Comparison of various values of the neutron-proton cross section, σ_{np}, at 158 MeV. The values in the column labeled "inelastic p-d" are obtained from the values of $\sigma_{np}+\sigma_{pp}$ given in Table II by subtracting the value of σ_{pp} measured by Caverzasio *et al.*[a] The values obtained from the phase-shift solutions of Breit *et al.*[b] and Kerman, McManus, and Thaler[c] are also given.

Laboratory angle	Inelastic p-d (mb)	YLAM YLAN3M (mb)	KMT (mb)
5°		7.92	6.42
10°	7.1 ±2.5	6.64	5.31
15°	4.84±0.44	5.51	4.42
20°	5.29±0.15	4.56	3.52

[a] See reference 19.
[b] See reference 10.
[c] See reference 2.

The values found for the parameters in Table II can be compared to those found from different considerations. For instance, in impulse approximation Σ_t is related to the elastic proton-deuteron center-of-mass cross section by[2]

$$d\sigma/d\Omega=(P_D/P_0)^2F^2(q)\Sigma_t, \tag{4.5}$$

where $F(q)$ is the deuteron form factor,

$$F(q)=\int|\phi_0|^2\exp(i\tfrac{1}{2}\mathbf{q}\cdot\mathbf{r})d\mathbf{r}, \tag{4.6}$$

and P_D is the momentum in the proton-deuteron center-of-mass system. The elastic cross section was also measured in SWC from which one obtains experimental values of Σ_t, using Eq. (4.5). In Table III these values of Σ_t are compared with those from Table II and also with the predictions of various nucleon-nucleon phase-shift solutions. The column labeled YLAM-YLAN3M gives the predictions of the phase-shift solutions of Breit and his collaborators[10] at 140 MeV. These are the best solutions of their energy-dependent phase-shift analysis.[17] The column labeled Prenowitz-YLAN3M is the prediction obtained by combining the $T=1$ phase shifts of Prenowitz[18] and the $T=0$ phase shifts (YLAN3M) of Breit. The column labeled KMT is the predictions using the nucleon-nucleon amplitudes given by Kerman, McManus and Thaler.[2] The inelastic and elastic values of Σ_t are seen to agree within the rather bad statistics. However these experimental values seem to be definitely smaller than the phase-shift predictions at 15° and 20°.

[17] The author wishes to thank Professor Breit for making available the relevant computer results.
[18] E. Prenowitz (private communication). The author wishes to thank E. Prenowitz for making his results available to him.

TABLE V. Comparison of the values of Σ_s found from the analysis of inelastic proton-deuteron scattering with the predictions of the phase-shift solutions of Breit *et al.*,[a] Prenowitz,[b] and Kerman, McManus, and Thaler.[c]

Laboratory angle	Inelastic *p-d* (mb)	YLAM YLAN3M (mb)	YLAM Prenowitz (mb)	KMT (mb)
5°	12.17±0.25	10.11	10.36	10.5
10°	11.50±0.70	8.02	7.78	7.59
15°	10.32±0.54	6.16	6.26	5.70
20°	6.70±0.45	4.96	5.43	4.91

[a] See reference 10.
[b] See reference 18.
[c] See reference 2.

The proton-proton cross section, σ_{pp}, has been measured at 158 MeV by Caverzasio *et al.*[19] at the relevant angles. Using these values and the values of $\sigma_{np}+\sigma_{pp}$ found in the present analysis, we get the values of σ_{np} given in Table IV. These values are compared with the values of σ_{np} predicted by the above phase-shift analyses. At 10° and 15° the values of σ_{np} found here are consistent with the phase-shift predictions. At 20° we find a value of σ_{np} larger than the predicted values, though the predicted values differ considerably between themselves. No reliable value of σ_{np} is obtainable at 5°.

There are no independent measurements of Σ_s. Table V compares the values of Σ_s found in the present analysis with the phase-shift predictions. In all cases the experimental values are considerably larger than the phase-shift values. A closer look at the amplitudes which go into Σ_s [Eq. (3.10)] shows that it is largely dominated by the $T=0$ amplitudes, particularly the real parts of B_0, E_0, and F_0. It is, of course, just the $T=0$ phase shifts for which we have the least information, resulting in significant statistical uncertainties in the phase-shift solutions.[20] It would be of great interest to see if the present $T=0$ phase-shift solution could be modified so as to fit the values of Σ_s found here, without doing violence to the rest of the neutron-proton data.

In conclusion it can be said that the small-angle slightly-inelastic proton-deuteron scattering data can be quantitatively understood in terms of the basic nucleon-nucleon interaction and neutron-proton final-state interaction in S states only. Three nucleon-nucleon parameters are needed to fit the data at a given angle. Two of these, Σ_t and $(\sigma_{pp}+\sigma_{np})$ are found to have values generally consistent with the values obtained from other considerations. However the most important parameter, Σ_s, is found to have a value much larger than that predicted from phase-shift solutions. The values of Σ_s found here may be of value in future searches for $T=0$ phase shifts.

ACKNOWLEDGMENTS

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[19] C. Caverzasio, K. Kuroda, and A. Michalowicz, J. Phys. Radium 21, 319 (1960).

[20] The second article in reference 10 lists the standard deviations of the $T=0$ phase shifts in solution YLAN3M. However, it is difficult to estimate the effect of such uncertainties on the predicted values of Σ_s, since there could be strong correlations among the phase shifts.